Reconstructing the Sanxian's Music Score by BD Matrix

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Outline

- Isospectral tridiagonal matrices.
- Extension to Hermitizable ones.
- Applications.

Tridiagonal/Birth-death matrix

$T \sim (a_k, -c_k, b_k), \quad E = \{k \in \mathbb{Z}_+ : 0 \leqslant k < N+1\}$

$$\begin{matrix} T \\ Q \end{matrix} = \begin{pmatrix} -c_0 & b_0 & & & 0 \\ a_1 & -c_1 & b_1 & & \\ & a_2 & -c_2 & b_2 & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & a_N & -c_N \end{pmatrix},$$

T: $(a_k), (b_k), (c_k)$ complex sequences.

Q: $a_k > 0$, $b_k > 0$, $c_k = a_k + b_k$, $c_N \geqslant a_N$.

Qin Dynasty (214 BC). Nanyin Trichord

Sanxian and Yis Sanxian Dance

Sanxian: three strings

Sound of each string, a Fourier series

Three sequence $\to$ tridiagonal matrix

Real spectrum $\to$ Hermitizable [selfadjoint]

Corollary

$\boldsymbol{T}$ is Hermitizable iff $(\boldsymbol{c_k})$ is real and

$\boldsymbol{a_{k+1} b_k > 0} [\Longleftrightarrow \boldsymbol{b_k / \bar{a}_{k+1} > 0}]\ \forall \boldsymbol{k}.$

$$\boldsymbol{\mu_k b_k = \mu_{k+1} \bar{a}_{k+1}}, \quad \boldsymbol{\mu_0 = 1}, \ \boldsymbol{\mu_{k+1} = \mu_k \frac{b_k}{\bar{a}_{k+1}}}.$$

Theorem/Algorithm

Given Hermitizable $T \sim (a_k, -c_k, b_k)$ with $c_k \geqslant |a_k| + |b_k|$ (or $\tilde{c}_k = c_k + m$) Then $\exists$ an explicit birth–death matrix $\tilde{Q} \sim (\tilde{a}_k, -\tilde{c}_k, \tilde{b}_k)$ such that T is isospectral to $\tilde{Q}$.

In general, we have $\tilde{c}_N \geqslant \tilde{a}_N$. We assume that $\tilde{c}_N > \tilde{a}_N$ in what follows. The case $\tilde{c}_N = \tilde{a}_N$ was also treated in the published paper (2018).

Explicit $u_k := a_k b_{k-1} > 0$ & $c_k =: \tilde{c}_k$

$$\tilde{b}_k = c_k - \cfrac{u_k}{c_{k-1} - \cfrac{u_{k-1}}{c_{k-2} - \cfrac{u_{k-2}}{\ddots\, c_2 - \cfrac{u_2}{c_1 - \cfrac{u_1}{c_0}}}}}$$

$$\tilde{b}_k = c_k - u_k / \tilde{b}_{k-1}, \quad \tilde{b}_0 = c_0$$

$$\tilde{a}_k = c_k - \tilde{b}_k, \quad k < N; \qquad \tilde{a}_N = u_N / \tilde{b}_{N-1}.$$

PF-a: $\tilde{b}_k > 0$. $c_k = |a_k| + |b_k|$. $u_k = a_k b_{k-1}$

$$\tilde{b}_0 = c_0 = |b_0| > 0,$$

$$\boxed{\tilde{b}_k = |b_k| > 0}$$

$$\tilde{b}_1 = c_1 - \frac{|a_1 b_0|}{\tilde{b}_0} = c_1 - \frac{|a_1 b_0|}{|b_0|} = |b_1| > 0,$$

$$\tilde{b}_2 = c_2 - \frac{|a_2 b_1|}{\tilde{b}_1} = c_2 - \frac{|a_2 b_1|}{|b_1|} = |b_2| > 0,$$

$$\cdots\cdots$$

$$\tilde{b}_{N-1} = c_{N-1} - \frac{|a_{N-1} b_{N-2}|}{b_{N-2}} = |b_{N-1}| > 0$$

if $N < \infty$.

PF-a: $\tilde{b}_k > 0$ if $c_k \geqslant |a_k| + |b_k|$.

Let $\bar{c}_0 > c_0$ and $c_k = (\geqslant) |a_k| + |b_k|$ for $k \neq 0$. Then $\bar{b}_0 = \bar{c}_0 > c_0 = \tilde{b}_0$. Furthermore, by induction, $\bar{b}_k > \tilde{b}_k$, $k \geqslant 1$.

In general, let **$(\tilde{b}_k)$ is increasing in (c_k)**

$$k_0 = \min\{k : c_k > |a_k| + |b_k|\}.$$

Then we have

$\bar{b}_k = \tilde{b}_k$, $0 \leqslant k \leqslant k_0 - 1$ and

$\bar{b}_k > \tilde{b}_k$, $k \geqslant k_0$.

PF-b: $\tilde{a}_k > 0,\ 1 \leqslant k < N + 1$

$$\tilde{a}_k = c_k - \tilde{b}_k,\ \ 1 \leqslant k < N$$

$$\tilde{b}_k = c_k - \tilde{a}_k = c_k - \frac{\tilde{a}_k \tilde{b}_{k-1}}{\tilde{b}_{k-1}}.$$

$$\tilde{b}_k = c_k - \frac{u_k}{\tilde{b}_{k-1}} = c_k - \frac{a_k b_{k-1}}{\tilde{b}_{k-1}}.$$

PF-b: $\tilde{a}_k > 0,\ 1 \leqslant k < N + 1.$

We obtain the invariant:

$$\tilde{a}_k \tilde{b}_{k-1} = a_k b_{k-1}, \quad 1 \leqslant k < N.$$

Equivalently,

$$\tilde{a}_k = u_k / \tilde{b}_{k-1} > 0, \quad 1 \leqslant k < N.$$

By def, this holds also for $k = N$ when $N < \infty$.

PF-c: T and $\tilde{Q}$ are isospectral

Define

Cost 4 years to get $\tilde{Q}$

$$h_0 = 1,\ h_{k+1} = h_k \frac{\tilde{b}_k}{b_k}, \quad 0 \leqslant k < N.$$

Then we have

C. & Xu Zhang, 2014

$$\tilde{Q} = \mathrm{Diag}(h)^{-1} T\, \mathrm{Diag}(h)$$

and so the conclusion follows.

Discrete spectrum. $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$

Let $\tilde{Q} \sim (\tilde{a}_k, -\tilde{c}_k, \tilde{b}_k)$. Define

$$\tilde{\mu}_0 = 1,\ \tilde{\mu}_k = \frac{\tilde{b}_0 \cdots \tilde{b}_{k-1}}{\tilde{a}_1 \cdots \tilde{a}_k}, \qquad k \geqslant 1.$$

Theorem (C. 2014. $\tilde{Q} \to T$)

(1) Let $\sum_{k=0}^{\infty} \left(\tilde{\mu}_k \tilde{b}_k\right)^{-1} < \infty$. Then $\text{Spec}\left(\tilde{Q}_{\min}\right)$ is discrete iff

$$\lim_{n\to\infty} \sum_{j=0}^{n} \tilde{\mu}_j \sum_{k=n}^{\infty} \left(\tilde{\mu}_k \tilde{b}_k\right)^{-1} = 0.$$

Discrete spectrum. $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$

Theorem (Continued)

(2) Let $\sum_{j=0}^{\infty} \tilde{\mu}_j < \infty$. Then Spec$(\widetilde{Q}_{\max})$ is discrete iff

$$\lim_{n\to\infty} \sum_{j=n+1}^{\infty} \tilde{\mu}_j \sum_{k=0}^{n} \left(\tilde{\mu}_k \tilde{b}_k\right)^{-1} = 0.$$

(3) Let $\sum_{k=0}^{\infty} \left(\tilde{\mu}_k \tilde{b}_k\right)^{-1} = \infty = \sum_{j=0}^{\infty} \tilde{\mu}_j$. Then Spec$(\widetilde{Q}_{\min})$ = Spec$(\widetilde{Q}_{\max})$ is not discrete.

$\sum_{i=0}^{\infty} \tilde{\mu}_i \sum_{j=i}^{\infty} \left(\tilde{\mu}_j \tilde{b}_j\right)^{-1} = \infty$

Hermitizable matrix

Definition

$A = (a_{ij})$ is Hermitizable if $\exists \mu > 0$ such that $\mu_i a_{ij} = \mu_j \bar{a}_{ji}$ $\forall i, j$.

Equivalently,

$$\text{Diag}(\mu) A = A^H \text{Diag}(\mu) \quad \boxed{A^H := \bar{A}^*}$$

Lemma

$A = (a_{ij})$ is Hermitizable iff $H := \text{Diag}(\mu)^{1/2} A \text{Diag}(\mu)^{-1/2}$ is Hermite.

Each theory/algorithm for Hermite $\to$ Hermitizable

Criterion for Hermitizability

Theorem (C. (2018))

Complex $A=(a_{ij})$ is Hermitizable iff two conditions hold simultaneously.

- For each pair i, j, either $a_{ij} \& a_{ji} = 0$ or $a_{ij}a_{ji} > 0$ $(\Leftrightarrow a_{ij}/\bar{a}_{ji} > 0)$.
- The circle condition holds for each smallest closed path without round-trip.

$$i_0 \to i_1 \to \cdots \to i_n = i_0,\ a_{i_k i_{k+1}} \neq 0$$

$$\Rightarrow a_{i_0 i_1} \cdots a_{i_{n-1} i_n} = \bar{a}_{i_n i_{n-1}} \cdots \bar{a}_{i_1 i_0}.$$

One

Computing the Hermitizing measure μ

Fix reference point i_0 and set $\mu_{i_0} = 1$.
For each $j \neq i_0$, choose and fix a path

$$i_0 \to i_1 \to \cdots \to i_n = j,$$

then

$$\mu_j = \frac{a_{i_0 i_1}}{\bar{a}_{i_1 i_0}} \frac{a_{i_1 i_2}}{\bar{a}_{i_2 i_1}} \cdots \frac{a_{i_{n-1} i_n}}{\bar{a}_{i_n i_{n-1}}}.$$

Circle condition $\Rightarrow$ path-independence.
Irreducible $\Rightarrow$ unique μ up to $+$constant, determined by $A = (a_{ij})$ only.

Householder transformation

Lemma

For each Hermite H, one can construct unitary U such that $T := UHU^H$ becomes a real, symmetric tridiagonal matrix. ★ $\Rightarrow$Hermitizable

Extended reflection matrices $\{U_j\}$:

$$U_j = I + (\kappa - 1)uu^H$$

κ: constant with $|\kappa| = 1$, u: unit vector.

$U := \prod_{j=0}^{\ell} U_j$ for some $\ell \leqslant N$.

History remark

Eigenproblem: started by C.G.J. Jacobi in 1846. 173 years

In 2000, two journals selected

"Top 10 algorithms in 20th century"

Three of them are on matrix eigenproblem,

Householder transformation

Any progress on eigenproblem is not easy!

Remove condition "tridiagonal"

Theorem (C. 2018)

Each irreducible Hermitizable matrix is isospectral to a birth–death Q-matrix.

Complex A: complex on $L^2(\mu)$

$\longrightarrow$ real BD $\tilde{Q}$: on real $L^2(|h|^2 \mathrm{d}x)$

Difference of h and $|h|$: the wave $e^{i\theta}$.

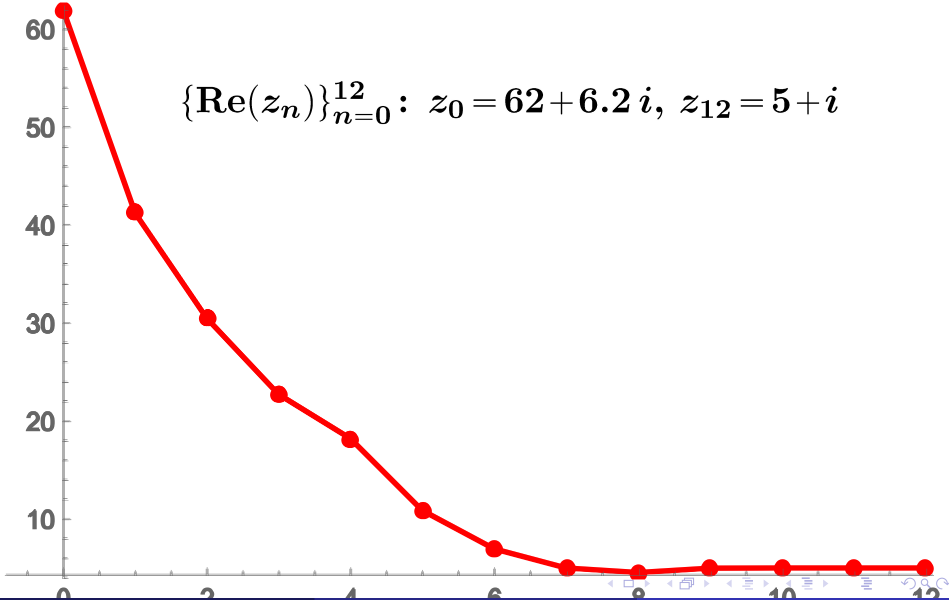
Quantum mechanics: wave-particle

Hermitizable $\longleftarrow$ Hermite $\longrightarrow$ BD-matrix

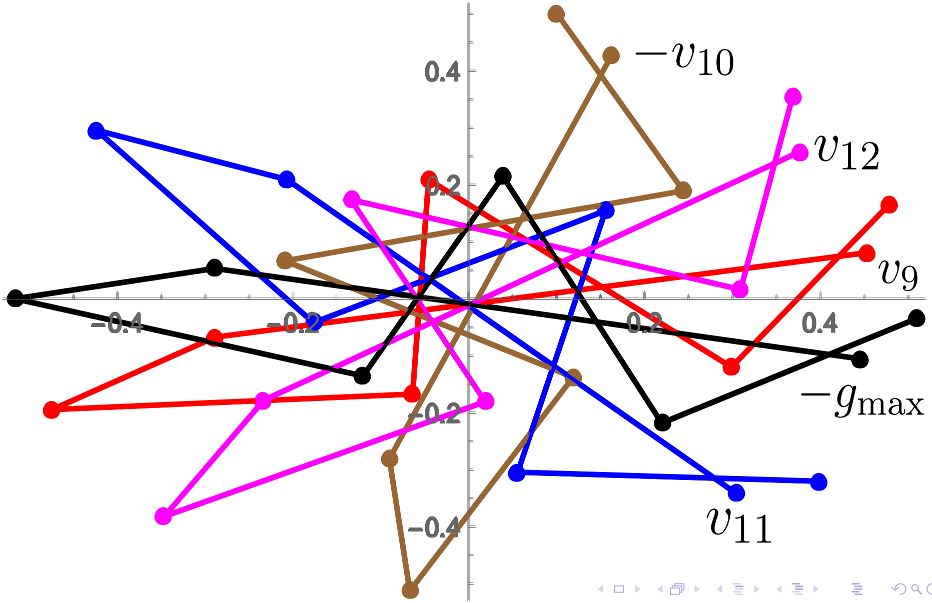
(non-uniform) (uniform media) (real)

Unified reference frame (spectrum)

$v_n \to g_{\max}$, $\mathrm{Re}(z_n) \to \max_j \mathrm{Re}(\lambda_j)$.

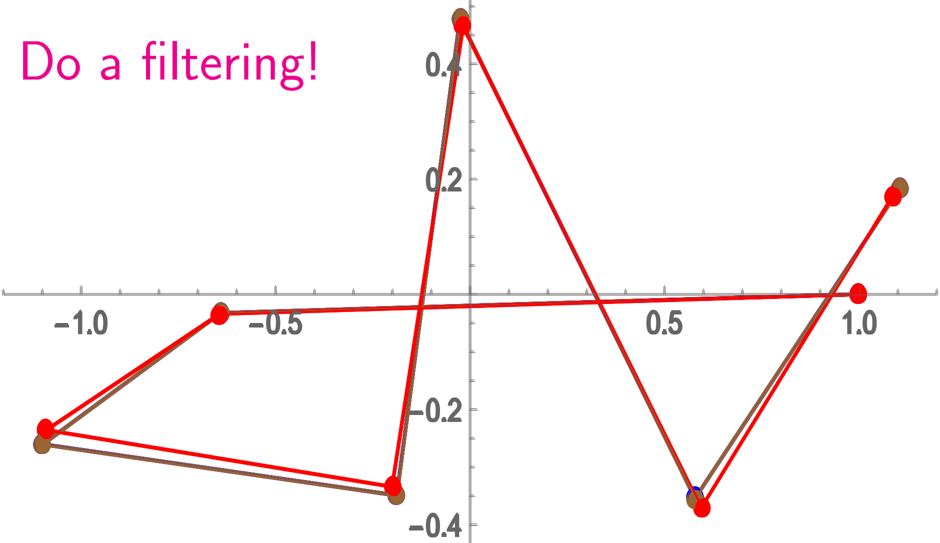


Vectors $\{v_n\}_{n=9}^{12}$ converge? 7-dim



$\|e^{i\theta_n}x\| = \|x\|$. $\tilde{v} := v/v(0)$. Conformal

Do a filtering!



$\tilde{v}_9$: red. $\tilde{v}_{10}$, $\tilde{v}_{11}$, $\tilde{v}_{12}$ and $\tilde{g}$ are coincided

Conclusions

Suggested in 20–30 minutes talk, use no more than 5 notation $\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{Q}, \boldsymbol{h}, \boldsymbol{u}$ and state no more than three main results:

- Isospectral of tridiagonal $\boldsymbol{T}$ and BD $\boldsymbol{Q}$.
- Criterion for Hermitizability.
- Isospectral of Hermitizable $\boldsymbol{A}$ and BD $\boldsymbol{Q}$.

http://math.bnu.edu.cn/~chenmf

The end!

Thank you, everybody!

谢谢大家！

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